Section 3.2: Feasible Sets

Linear Programming problems often have several constraints, leading to several inequalities or a **system of linear inequalities**. A point (x, y) **satisfies a system of inequalities** if it satisfies all of the inequalities in the system.

The **solution set of a system of linear inequalities** is the set of all points in the plane which satisfy the system of inequalities. This is also called the **feasible set** of the system of inequalities or the **feasible region** of the system.

The graph of the feasible set for a system of inequalities is the set of all points in intersection of the graphs of the individual inequalities.

Example Determine if $(x, y) = (1, 2)$ is in the feasible set for the system of inequalities shown below and graph the feasible set for the system of inequalities:

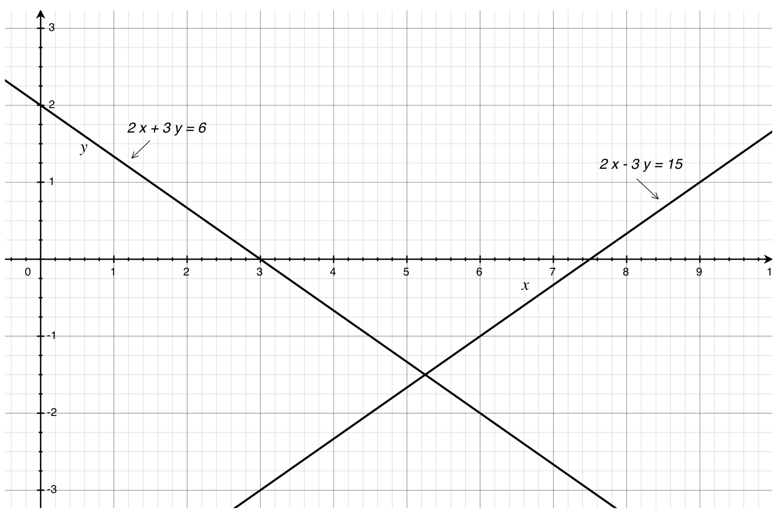
$$2x + 3y \geq 6$$

$$2x - 3y \geq 15$$

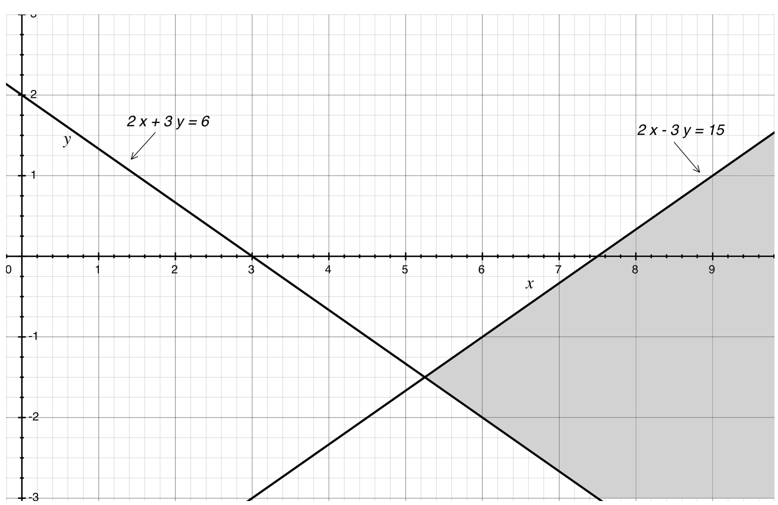
$$2 \cdot 1 + 3 \cdot 2 = 8 \geq 6$$
$$2 \cdot 1 - 3 \cdot 2 = -4 \not\geq 15$$

so $(1, 2)$ is not in the feasibility set.

The two lines:



The feasible set is shaded:



In general, to sketch the feasible set/region for a system of inequalities :

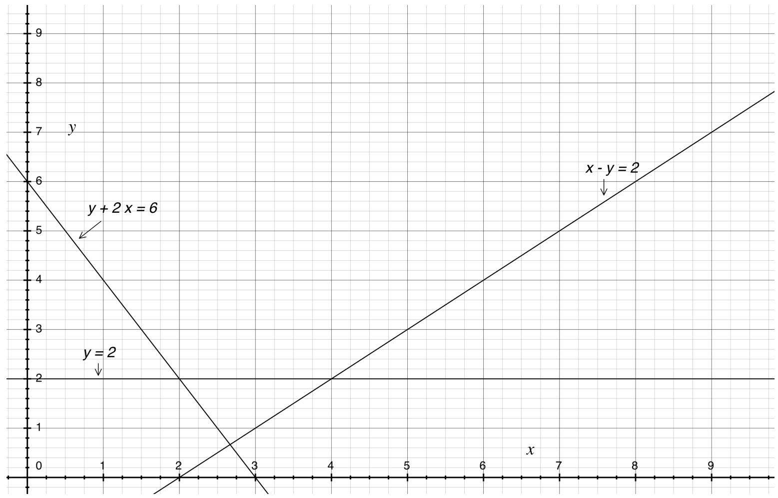
- Replace each inequality symbol with an equals sign to obtain a linear equation.
- Graph each line. Use a solid line if it is part of the solution (when $\leq$ or $\geq$ are used), use a dotted line if it is not part of the solution (when $<$ or $>$ are used).
- Select a test point not on the line.
- If the test point satisfies the inequality the graph of the inequality is on that side of the line, otherwise it is on the opposite side.
- Shade the side corresponding to the inequality or use arrows to indicate which side it is.
- The region where all shadings overlap is the feasible set.

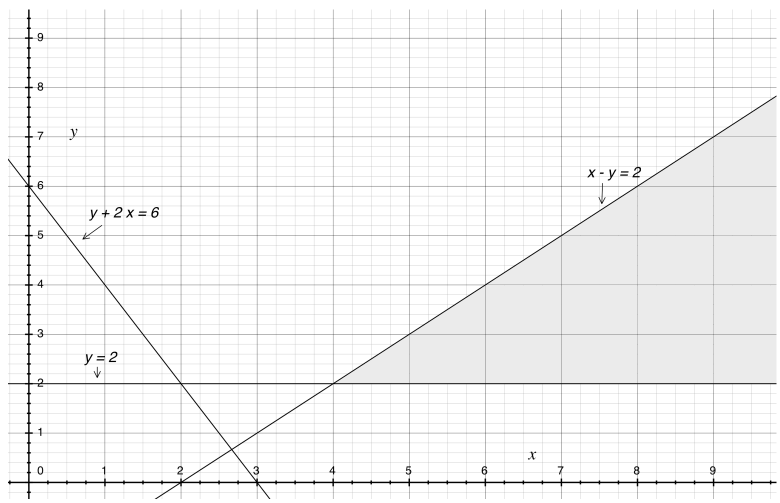
Example Graph the feasible set for the system of inequalities:

$$x - y \geq 2$$

$$y + 2x \geq 6$$

$$y \geq 2$$

The three lines: 

The feasible set is shaded: 

A feasible set is **bounded** if it can be contained in some square centered at the origin (with sides of finite length). Otherwise the feasible set is **unbounded**.

The feasible set in the first example above is unbounded and the feasible set in the second example is bounded.

The boundaries of the feasible set of a system of linear inequalities will be parts of the graphs of some of the associated linear equalities(lines). The **corners** or **vertices** of the feasible set will be points at which these lines intersect. We will need to find the co-ordinates of the vertices of such a feasible set to solve the linear programming problems in the next section.

Intersection of a pair of lines

An easy way to find the intersection of a pair of lines (both non vertical), is to rearrange their equation to the (standard) form shown below and equate y values;

$$y = m_1 x + b_1 \text{ and } y = m_2 x + b_2$$

intersect where

$$m_1 x + b_1 = m_2 x + b_2.$$

Example Find the point of intersection of the lines:

$$2x + 3y = 6$$

$$2x - 3y = 15$$

$$\begin{aligned} y &= -\frac{2}{3}x + 2 \\ y &= \ \ \frac{2}{3}x - 5 \end{aligned}$$

so $\frac{2}{3}x - 5 = -\frac{2}{3}x + 2$ or $\frac{4}{3}x = 2 + 5$. Then $4x = 3 \cdot (7) = 21$ so $x = \frac{21}{4}$. Then $y = \frac{2}{3}\left(\frac{21}{4}\right) - 5 = \frac{7}{2} - 5 = -\frac{3}{2}$. So $\left(\frac{21}{4}, -\frac{3}{2}\right)$ is the point of intersection.

To find the vertices/corners of the feasible set, graph the feasible set and identify which lines intersect at the corners. Use the graphs you drew above to solve the problems below.

Example Find the Vertices of the feasible set corresponding to the system of inequalities:

$$2x + 3y \geq 6$$

$$2x - 3y \geq 15$$

This is the same problem we just worked. The two lines are not parallel or equal so they intersect in one point, $\left(\frac{21}{4}, -\frac{3}{2}\right)$.

Example Find the Vertices of the feasible set corresponding to the system of inequalities:

$$x - y \geq 2$$

$$y + 2x \geq 6$$

$$y \geq 2$$

No two of these three lines are parallel or equal so there are three vertices.

$x - y = 2$ and $y + 2x = 6$ intersect as follows: $y = x - 2$, $y = -2x + 6$ so $x - 2 = -2x + 6$ or $3x = 6 + 2$ so $x = \frac{8}{3}$ and then $y = \frac{8}{3} - 2 = \frac{2}{3}$ so the intersection is $\left(\dfrac{8}{3}, \dfrac{2}{3}\right)$.

$x - y = 2$ and $y = 2$ intersect as follows: $y = x - 2$, $y = 2$ so $x - 2 = 2$ or $x = 4$ and then $y = 2$ so the intersection is $(4, 2)$.

$y = 2$ and $y + 2x = 6$ intersect as follows: $y = 2$, $y = -2x + 6$ so $2 = -2x + 6$ or $x = 2$ and then $y = 2$ so the intersection is $(2, 2)$.

There is only one vertex in the feasible set, $(4, 2)$.

Empty Feasible Sets

Sometimes there are no points in the feasible set for a system of inequalities as in the following example.

Example Graph the feasible set for the system of inequalities:

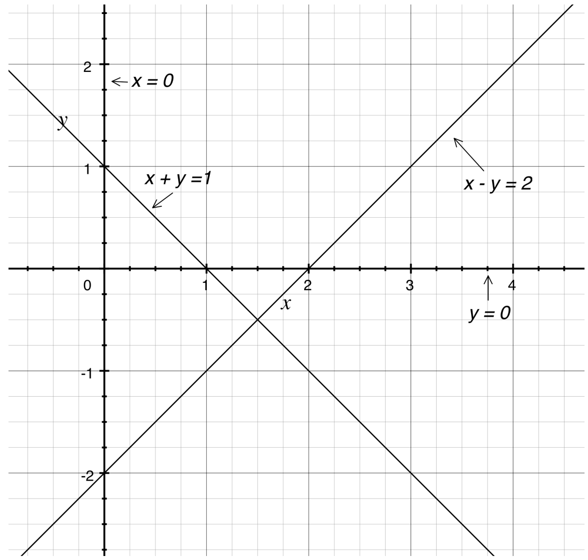
$$x - y \geq 2$$

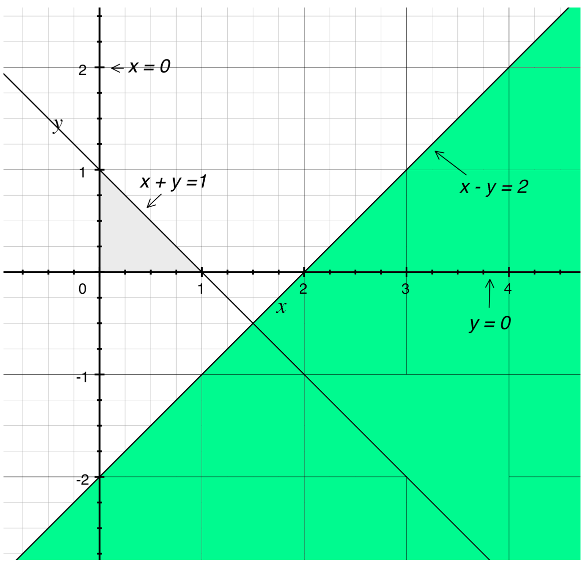
$$x + y \leq 1$$

$$y \geq 0$$

$$x \geq 0$$

The two lines which are not the axes are





The gray area is the feasible set for

$$x + y \leq 1$$

$$y \geq 0$$

$$x \geq 0$$

and the green area is the feasible set for $x - y \geq 2$. The feasible set is the intersection and hence is empty.

Setting up the inequalities

Example Mr. Carter eats a mix of Cereal A and Cereal B for breakfast. The amount of calories and sodium per ounce for each is shown in the table below. Mr. Carter's breakfast should provide at least 480 calories but less than 700 milligrams of sodium.

	Cereal A	**Cereal B**
Calories(per ounce)	100	140
Sodium(mg per ounce)	150	190

Let x denote the number of ounces of Cereal A that Mr. Carter has for breakfast and let y denote the number of ounces of Cereal B that Mr. Carter has for breakfast. What are the set of constraints on the amounts of each cereal that Mr. Carter can consume for breakfast.

$$\begin{aligned} 100x + 140y &\geqslant 480 \\ 150x + 190y &< 700 \\ x &\geqslant 0 \\ y &\geqslant 0 \end{aligned}$$

Example A juice stand sells two types of fresh juice in 12 oz cups. The Refresher and the Super Duper. The Refresher is made from 3 oranges, 2 apples and a slice of ginger. The Super Duper is made from one slice of watermelon and 3 apples and one orange. The owners of the juice stand have 50 oranges, 40 apples, 10 slices of watermelon and 15 slices of ginger. Let x denote the number of Refreshers they make and let y denote the number of Super Dupers they make. What is the set of constraints on x and y?

$$
\begin{aligned}
&2x + 3y \leqslant 40 \quad \textit{Apples}\\
&3x + 1y \leqslant 50 \quad \textit{Oranges}\\
&0x + 1y \leqslant 10 \quad \textit{Watermelon}\\
&1x + 0y \leqslant 15 \quad \textit{Ginger}\\
&x \geqslant 0\\
&y \geqslant 0
\end{aligned}
$$

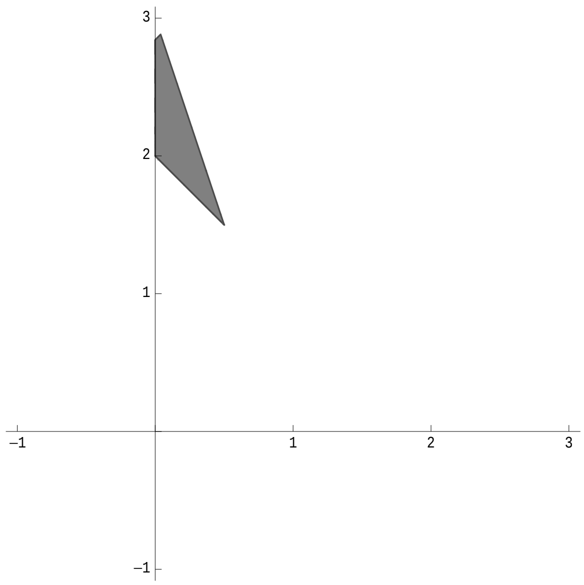
Note for many of the old exam questions the shading is opposite to that shown above (The unshaded region denotes the feasible set)

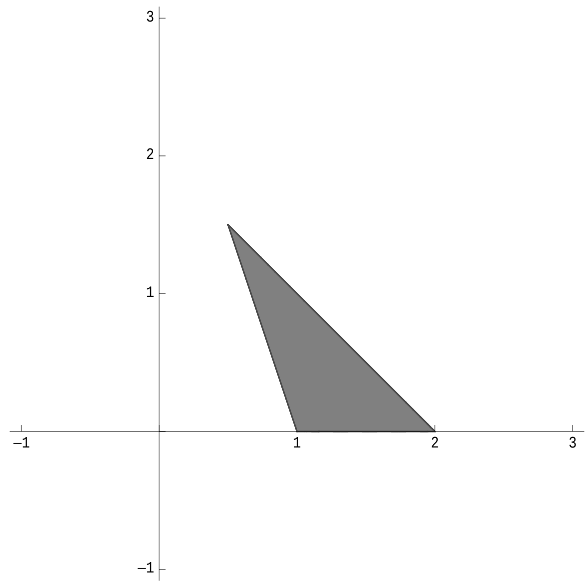
Extras: Old Exam Questions

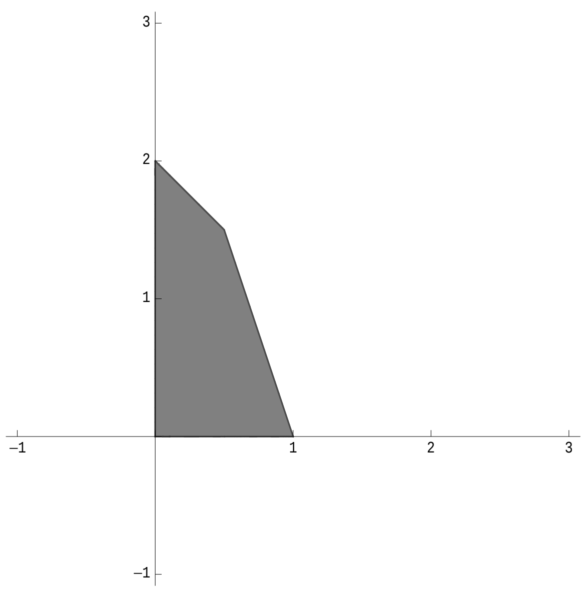
1 Select the graph of the feasible set of the system of linear inequalities given by:

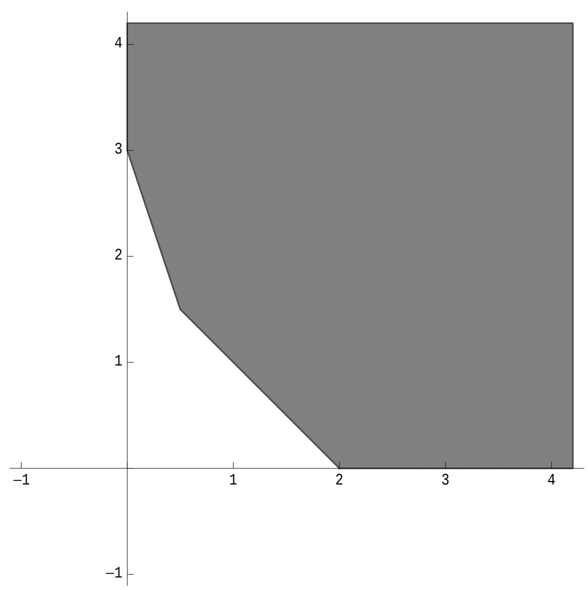
$$\begin{array}{rcrcl} & & x & \geq & 0 \\ & & y & \geq & 0 \\ 3x & + & y & \leq & 3 \\ 2x & + & 2y & \leq & 4 \end{array}$$

where the shaded area is the feasible set.



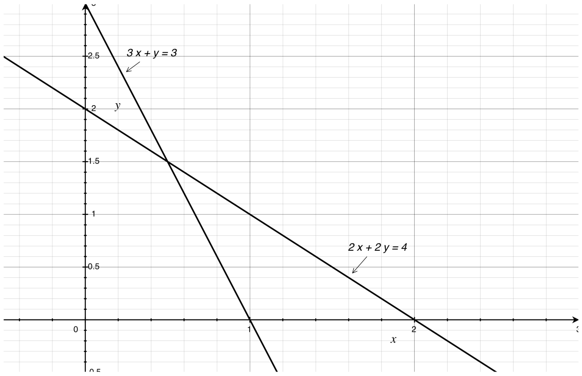




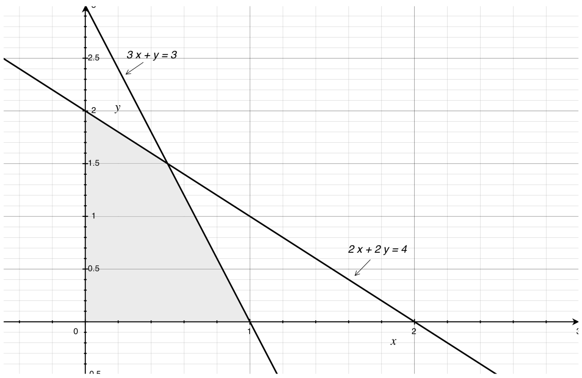


A quick solution is to note that $(0, 0)$ satisfies all the inequalities. Hence the lower left is the only possible answer.

Or just draw the lines and shade the feasible set. The lines are



and the feasible set is



2 A student spending spring break in Ireland wants to visit Galway and Cork. The student has at most 7 days available and at most 500 euros to spend. Each day spent in Galway will cost 50 euros and each day spent in Cork will cost 60 euros. Let x be the number of days the student will spend in Galway and y, the number of days the student will spend in Cork. Which of the following sets of constraints describe the constraints on the student's time and money for the visits?

$$(a) \quad \begin{array}{c} x + y \leq 7 \\ 50x + 60y \leq 500 \\ x \geq 0, \quad y \geq 0 \end{array} \qquad (b) \quad \begin{array}{c} x + 7y \leq 500 \\ 50x + 60y \leq 1000 \\ x \geq 0, \quad y \geq 0 \end{array} \qquad (c) \quad \begin{array}{c} x + y \leq 7 \\ 60x + 50y \leq 500 \\ x \geq 0, \quad y \geq 0 \end{array}$$

$$(d) \quad \begin{array}{c} x + y \geq 7 \\ 50x + 60y \geq 500 \\ x \geq 0, \quad y \geq 0 \end{array} \qquad (e) \quad \begin{array}{c} x + y \geq 7 \\ 60x + 50y \geq 500 \\ x \geq 0, \quad y \geq 0 \end{array}$$

$$\begin{array}{rcrcrl} x & + & y & \leqslant & 7 & \text{Days} \\ 50x & + & 60y & \leqslant & 500 & \text{euros} \\ x \geqslant 0 & & y \geqslant 0 & & & \end{array}$$